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LETTER TO THE EDITOR

Deconstructing an integrable lattice equation

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Abstract
We show that an integrable lattice equation, obtained by J Hietarinta using the 'consistency around a cube' method without the tetrahedron assumption, is indeed solvable by linearization. We also present its nonautonomous extension.

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1. Introduction

Constructing integrable lattice equations is a procedure which is both delicate and not necessarily systematic. Two main approaches have been successfully used over the years. The first one is indeed constructive [1]. One starts from some spectral problem, introducing a Lax pair (usually, a family thereof) and derives all integrable equations which are associated with this linear problem. While this method has the advantage of leading to systems which are integrable by construction, it is not useful when it comes to dealing with lattice equations which are obtained from some physical models and are thus given *a priori*. The second method is tailored so as to deal with the latter situation. It is based on detection rather than construction. Given a lattice equation, one applies one (or, better, several) integrability detectors [2, 3] and if the system satisfies all integrability criteria one can confidently postulate its integrability. The drawback of this method lies, of course, in the fact that it does not provide a proof of integrability and one must in principle complement the study of the system by actually integrating it.

Integrable lattice equations are particularly interesting. They contain and extend the integrable (differential) evolution equations which can be obtained as (continuous) limits form the discrete ones. Thus, the integrable lattice equations provide ideally suitable *integrable* integrators for the corresponding evolution equations. Discrete systems on lattices can play an important role in their understanding of discrete integrability. As a matter of fact the singularity confinement property was discovered [2] by studying the lattice KdV equation. Moreover one-dimensional mappings, such as the discrete Painlevé equations or discrete

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many-body systems, can be viewed as reductions of integrable lattices. Connections of the latter to differential geometry and the combinatorics of graphs are also known to exist.

While integration with spectral methods, based on the existence of Lax pairs is the usual way of integrating a given integrable evolution or lattice equation, there exists another type of integrability (often ignored). Calogero [4] has coined the term c-integrability for the latter, and what he means by this is that the system can be reduced to a linear one through the help of a Cole–Hopf-type transformation. Thus, while integrability with the spectral method of an evolution equation (s-integrability in the sense of Calogero) leads to a linear integrodifferential system, in the case of c-integrability the linear system is a purely differential one. As we have shown in [5] integrability through linearization does not require the Painlevé property (for differential systems) or singularity confinement (for discrete ones). The aim of the present paper is to analyse a system obtained by Hietarinta [6] who has proposed its integrability through Lax pairs. We shall show that the system is much simpler than what was implied and provide its explicit reduction to a linear equation.

In [6], Hietarinta has examined critically the work of Adler, Bobenko and Suris [7] who have generated families of integrable lattice equations based on the 'consistency around a cube' (CAC) approach [8]. The main idea of this method is the following. One starts from a two-dimensional square lattice, define the variable on the vertices $x_{n,m}$, $x_{n,m+1}$, $x_{n+1,m}$, $x_{n+1,m+1}$ and write the *multilinear* equation relating these variables. In this way, solving for $x_{n+1,m+1}$ gives a rational expression of the other x's. For the CAC trick, one adjoins a third direction, say k, and imagine the mapping giving $x_{n+1,m+1,k+1}$ as being the composition of mappings on the various planes. There exist three different ways to obtain $x_{n+1,m+1,k+1}$ and the consistency requirement is that they lead to the same result. This places severe constraints on the multilinear equation, but they do not suffice to determine it completely. Adler, Bobenko and Suris have introduced two additional assumptions. They considered only a certain class of symmetrical forms for the multilinear equation and also they required that $x_{n+1,m+1,k+1}$ be independent of $x_{n,m,k}$ (the tetrahedron property). Under the constraints of these simplifying assumptions, they were able to produce a complete classification of lattice systems. The latter are all integrable, since the procedure also furnishes their Lax pairs.

Hietarinta questioned these assumptions and produced one integrable lattice equation which did not make use of the tetrahedron property. He also obtained the Lax pair for this system, but as we shall show in the following section the integrability of this lattice equation is of a much simpler type.

2. Hietarinta's lattice equation

The lattice equation of Hietarinta has the form:

$$\frac{x_{n,m}+b}{x_{n,m}+a}\,\frac{x_{n+1,m+1}+d}{x_{n+1,m+1}+c}=\frac{x_{n+1,m}+b}{x_{n+1,m}+c}\,\frac{x_{n,m+1}+d}{x_{n,m+1}+a}. \tag{1}$$

Our first approach to equation (1) is, in the ARS [9] spirit, through its reductions. The simplest nontrivial reduction of (1) is obtained from the periodicity condition $x_{n+1,m+1}=x_{n,m}$. We readily find the mapping (omitting the second index)

$$\frac{x_{n+1}+b}{x_{n+1}+c}\,\frac{x_{n-1}+d}{x_{n-1}+a}=\frac{x_n+b}{x_n+a}\,\frac{x_n+d}{x_n+c}. \tag{2}$$

Next, we study the integrability of this equation using the algebraic entropy method [3]. For this we compute the degree growth of the numerator and denominator of the iterates of (2) in terms of the initial conditions. We find that the degree of the nth iterate grows like $d_n=n$. According to our results in [10] not only is (2) integrable, but moreover it is a linearizable mapping. An elementary calculation shows that (2) can be reduced to the homographic mapping

$$x_{n+1} = \frac{x_n((bd-ac)+(c-b)k)+ad(b-c)+(cd-ab)k}{(d-a)(x_n+k)}, \tag{3}$$

where k is an integration constant. This is a highly nontrivial result and it leads to the hypothesis that the full system (1) might be linearizable. This turns out to be the case indeed. As a complementary indication we have computed the degree growth of the iterates of the mapping (1) and we found again linear growth, reinforcing thus the linearizability hypothesis.

In order to linearize the lattice we start by a homographic transformation. We set

$$x_{n,m} = \frac{d-cy_{n,m}}{y_{n,m}-1} \tag{4}$$

and obtain the lattice equation

$$y_{n+1,m+1}(y_{n,m}+B)(y_{n,m+1}+A) = y_{n,m+1}(y_{n,m}+A)(y_{n+1,m}+B), \tag{5}$$

where $B = (d-b)/(b-c)$, $A = (d-a)/(a-c)$. Next, we remark that (5) can be put in the form

$$\frac{V_{n+1,m}}{V_{n,m}} = \frac{U_{n,}}{U_{n,m+1}}, \tag{6}$$

where $V_{n,m} = y_{n,m+1}/(y_{n,m}+B)$ and $U_{n,m} = y_{n,m}+A$. Equation (6) in turn can be parametrized by $V_{n,m} = Q_{n,m}/Q_{n,m+1}$. Introducing $W_{n,m} = Q_{n+1,m}/Q_{n,m}$, we find $W_{n,m}/W_{n,m+1} = U_{n,m}/U_{n,m+1}$. The solution of the latter is $U_{n,m} = f(n)W_{n,m}$, with f a free function of n, leading to $y_{n,m}+A = f(n)Q_{n+1,m}/Q_{n,m}$ and $y_{n,m+1}/(y_{n,m}+B) = Q_{n,m}/Q_{n,m+1}$. Since only $f(n)$ appears in the definition of y through a term $f(n)Q_{n+1,m}/Q_{n,m}$, it is clear that it can be absorbed by a gauge transformation of Q and thus we can simply omit it. Eliminating y, we obtain finally

$$Q_{n+1,m+1} - AQ_{n,m+1} - Q_{n+1,m} + (A-B)Q_{n,m} = 0, \tag{7}$$

i.e. a linear equation for Q. Solving (7) we can obtain y and thus reconstitute x.

The reduction of the lattice equation (1) to the form (5) was very helpful for its linearization. It is interesting to note that an equation of the form (5) can be obtained as a limit of (1). Indeed, taking $d \to 0, c \to \infty$ we find

$$x_{n+1,m+1}(x_{n,m}+b)(x_{n,m+1}+a) = x_{n,m+1}(x_{n,m}+a)(x_{n+1,m}+b) \tag{8}$$

which is exactly (5) with $y = x$, $A = a$, $B = b$.

At this point one may wonder what is the consequence of the linearizability on the Lax pair. In [6], Hieratinta has given the Lax pair for (1). In order to simplify the presentation, we analyse below the Lax pair for equation (8), which is obtained from that of (1) by taking $d \to 0, c \to \infty$. This does not change anything in the argument, but does simplify the computations. We have

$$\begin{aligned} L_1(n,m) &= \begin{pmatrix} \dfrac{\mu-a}{x_{n,m}+a} & \dfrac{\mu a}{x_{n,m}+a} - \dfrac{\mu x_{n+1,m}}{x_{n,m}} \\ 0 & -\dfrac{x_{n+1,m}}{x_{n,m}} \end{pmatrix} \\ L_2(n,m) &= \begin{pmatrix} \dfrac{(\mu-b)x_{n,m+1}}{x_{n,m}+b} & \dfrac{\mu b x_{n,m+1}}{x_{n,m}+b} \\ 0 & \dfrac{\mu x_{n,m+1}}{x_{n,m}} \end{pmatrix} \end{aligned} \tag{9}$$

and the lattice equation is obtained from the compatibility relation

$$L_2(n, m+1)L_1(n, m) = L_1(n+1, m)L_2(n, m). \tag{10}$$

Since the lattice equation is independent of the spectral parameters λ, μ, we have simplified further L_1 and L_2 by taking $\lambda = 0$. Next, we introduce the expression of x in terms of the quantity appearing in the linear equation, $x_{n,m} = Q_{n+1,m}/Q_{n,m} - a$ and obtain for Q the equation

$$\begin{aligned} &Q_{n+2,m+1}(Q_{n+1,m} + (b-a)Q_{n,m}) - Q_{n+1,m+1}(Q_{n+2,m} + bQ_{n+1,m} + a(b-a)Q_{n,m}) \\ &\quad + aQ_{n,m+1}(Q_{n+2,m} + (b-a)Q_{n+1,m}) = 0. \end{aligned} \tag{11}$$

It is then straightforward to show that (11) is a consequence of the linear equation $Q_{n+1,m+1} - aQ_{n,m+1} - Q_{n+1,m} + (a-b)Q_{n,m} = 0$.

Finally, one can wonder what are the (integrable) nonautonomous forms of the lattice equation (1). The answer to this question is straightforward if we start from the linear equation for Q

$$Q_{n+1,m+1} + f_{n,m}Q_{n,m+1} + g_{n,m}Q_{n+1,m} + h_{n,m}Q_{n,m} = 0. \tag{12}$$

We assume now that f, g and h are free functions of n and m. Next, we introduce the nonlinear variable $x_{n,m} = Q_{n,m+1}/Q_{n,m}$ and upshift (12) in the m direction and eliminate Q. We thus find

$$(x_{n+1,m+1} + f_{n,m+1})x_{n+1,m}(g_{n,m}x_{n,m} + h_{n,m}) = (x_{n+1,m} + f_{n,m})x_{n,m}(g_{n,m+1}x_{n,m+1} + h_{n,m+1}). \tag{13}$$

In order to bring (13) under the form of (1), we may perform a homographic transformation consisting in a translation, an inversion of the dependent variable followed by a new translation. A total freedom exists at the homography level and two new free functions can thus be introduced leading to the most general nonautonomous form of (1).

3. Conclusion

In this paper we have presented what we called the deconstruction of a lattice equation. Namely, starting from a given equation, the integrability of which was already established, we decided to probe deeper and thus discovered that its integrability was of a simpler nature than the one implied in the original paper. This raises an interesting question: is this an exceptional feature pertinent only to the equation at hand or is this a common feature for the equations derived by the CAC principle without the tetrahedron assumption? Clearly, the analysis of more examples is necessary before this question can be settled.

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